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From 2D integrable systems to self-dual gravity

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Abstract. We explain how to construct solutions to the self-dual Einstein vacuum equations from solutions of various two-dimensional (2D) integrable systems by exploiting the fact that the Lax formulations of both systems can be embedded in that of the self-dual Yang–Mills equations. We illustrate this by constructing explicit self-dual vacuum metrics on $\mathbb{R}^2 \times \Sigma$, where Σ is a homogeneous space for a real subgroup of $SL(2, \mathbb{C})$ associated with the 2D system.

1. Introduction

Ward [12] has observed that many integrable systems, particularly in two dimensions, may be obtained from the self-dual Yang–Mills (SDYM) equations by symmetry reduction. See [2] for a survey of such reductions. See also [8] for an account of how reductions can be used as a framework for classification, and for a survey of the applications of twistor theory.

It has been shown [5] that the SDYM equations with volume-preserving diffeomorphisms SDiff($\mathcal{M}$) of a four-manifold $\mathcal{M}$ gauge group and translational symmetry in all four variables reduces to the self-dual (SD) Einstein vacuum equations on $\mathcal{M}$. This result extends the work of Ashtekar *et al* [1]. It also implies, [13], that solutions of the SDYM equations with two translational symmetries and gauge group SDiff(Σ) for some two-manifold Σ also determine the solutions of the SD Einstein vacuum equations.

The aim of the present paper is to show that the correspondence between the Lax formulations of certain two-dimensional (2D) integrable systems and the SD Einstein equations enables us to construct SD vacuum metrics explicitly from solutions to various 2D nonlinear integrable equations. We do this by considering $SL(2, \mathbb{C})$ SDYM fields invariant under the action of two translations of spacetime. These fields can be represented as solutions of various soliton equations in two dimensions. Self-dual vacuum metrics are recovered by representing the Lie algebra of (real forms of) $SL(2, \mathbb{C})$ as Hamiltonian vector fields on a 2D homogeneous space for the gauge group.

Other approaches to self-dual gravity that reveal its connection with 2D integrable systems have been given by Ward [13] and Q-Han Park [9].

In the next section we review briefly the classification of 2D integrable systems arising from the $SL(2, \mathbb{C})$ SDYM equations. In section 3 we discuss the connection between the SDYM equations and self-dual gravity. Section 4 is devoted to the construction of normalized null-tetrads and hence metrics on $\mathbb{R}^2 \times \Sigma$ from the SDYM Lax pairs for the 2D integrable systems. In the last section we outline the twistor interpretation of the construction.

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2. Self-dual Yang–Mills and 2D integrable systems

Consider a Yang–Mills vector bundle over a four-dimensional manifold $\mathbb{M}$ (taken here to be $\mathbb{C}^4$ in general, or $\mathbb{R}^4$ when reality conditions are imposed) with connection one-form $A = A_\mu(x^\nu)\,\mathrm{d}x^\mu \in T^*\mathbb{M} \otimes LG$, where LG is the Lie algebra of some gauge group G. The corresponding curvature $F = F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is given by

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu] \tag{2.1}$$

where $D_\mu = \partial_\mu - A_\mu$ is the covariant derivative. The SDYM equations on a connection A are the self-duality conditions on the curvature under the Hodge star operation

$$F = *F \qquad \text{or in index notation} \qquad F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}. \tag{2.2}$$

They are conformally invariant and are also preserved by the gauge transformations

$$A \to g^{-1}Ag - g^{-1}\,\mathrm{d}g \qquad g \in \mathrm{Map}(\mathbb{M}, G). \tag{2.3}$$

Let us introduce the double-null coordinates w, $\tilde{w}$, z and $\tilde{z}$, in which the metric on $\mathbb{M}$ is $\mathrm{d}s^2 = \mathrm{d}w\,\mathrm{d}\tilde{w} - \mathrm{d}z\,\mathrm{d}\tilde{z}$. In these coordinates the SDYM equations may be rewritten as

$$F_{wz} = 0 \tag{2.4}$$

$$F_{\tilde{w}\tilde{z}} = 0 \tag{2.5}$$

$$F_{w\tilde{w}} - F_{z\tilde{z}} = 0 \tag{2.6}$$

which are the compatibility conditions $[L, M] = 0$ for the linear system of equations $L\Phi = 0$, $M\Phi = 0$ where the 'Lax pair', L and M, are

$$L = D_w - \lambda D_{\tilde{z}} \qquad M = D_z - \lambda D_{\tilde{w}} \tag{2.7}$$

for $\lambda \in \mathbb{C}P^1$ and Φ is an n-component column vector.

We shall consider the reality conditions for real ultra-hyperbolic spaces, recovered by imposing $w = x - y$, $z = t + v$, $\tilde{w} = x + y$ and $\tilde{z} = v - t$. (Reality conditions for Euclidean space are recovered by imposing $\tilde{w} = \bar{w}$ and $\tilde{z} = -\bar{z}$.) Solutions to (2.4)–(2.6) can be real for this choice of signature.

We fix the gauge group to be $SL(2, \mathbb{C})$ or one of its real subgroups. Conformal reduction of the SDYM equations involves the choice of the group H of conformal isometries of $\mathbb{M}$. We shall restrict ourselves to the simplest case and suppose that a connection A is invariant under the flows of two independent translational Killing vectors X and Y. These reductions are classified partially by the signature of the metric restricted to the two-plane spanned by the translations.

2.1. Non-degenerate cases (H_1)

(a) $X = \partial_w - \partial_{\tilde{w}}$, $Y = \partial_z - \partial_{\tilde{z}}$.

$$A_w = \frac{1}{4}\begin{pmatrix} \phi_t & -2\cos(\phi/2) \\ -2\cos(\phi/2) & -\phi_t \end{pmatrix} \qquad A_{\tilde{w}} = \frac{1}{4}\begin{pmatrix} \phi_t & 2\cos(\phi/2) \\ 2\cos(\phi/2) & -\phi_t \end{pmatrix}$$

$$A_z = \frac{1}{4}\begin{pmatrix} -\phi_x & -2\sin(\phi/2) \\ 2\sin(\phi/2) & \phi_x \end{pmatrix} \qquad A_{\tilde{z}} = \frac{1}{4}\begin{pmatrix} -\phi_x & 2\sin(\phi/2) \\ -2\sin(\phi/2) & \phi_x \end{pmatrix}. \tag{2.8}$$

The SDYM equations are satisfied in ultra-hyperbolic signature if $\phi_{xx} + \phi_{tt} = \sin\phi$; the elliptic sine–Gordon equation.

(b) $G = SU(2)$, $X = \partial_w$, $Y = \partial_{\tilde{w}}$.

$$A_{\tilde{z}} = 0 \qquad A_w = \cos\phi \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$A_{\tilde{w}} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad A_z = 1/2(\phi_v - \phi_t) \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{2.9}$$

The SDYM equations in ultra-hyperbolic signature yield $\phi_{tt} - \phi_{vv} = 4\sin\phi$, the hyperbolic sine–Gordon equation.

For details of these reductions, see [8, ch 6] or [2]. Note also that if we reduce from Euclidean signature we obtain Hitchin's Higgs bundle equations (which can also be represented as harmonic maps from $\mathbb{R}^2$ to $SL(2, \mathbb{C})/G$ where G is $SU(2)$ or $SU(1, 1)$) [8].

2.2. *Partially degenerate case (H_2)*

We consider the ultra-hyperbolic signature only with $X = \partial_w - \partial_{\tilde{w}}$ and $Y = \partial_{\tilde{z}}$.

(a)

$$A_w = \begin{pmatrix} q & 1 \\ b & -q \end{pmatrix} \qquad A_{\tilde{w}} = 0$$
$$2A_z = \begin{pmatrix} b_x & -2q_x \\ 2w & -b_x \end{pmatrix} \qquad A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{2.10}$$

where $4w = q_{xxx} - 4qq_x - 2q_x^2 + 4q^2 q_x$ and $b = q_x - q^2$. The SDYM equations (with the definition $u = -q_x$) are equivalent to the Korteweg de Vries (KdV) equation $4u_z = u_{xxx} + 12uu_x$. The reduced Lax pair (2.7) yields the standard zero curvature representation of the KdV equation [14].

(b)

$$A_w = \begin{pmatrix} 0 & \phi \\ \pm\bar{\phi} & 0 \end{pmatrix} \qquad A_{\tilde{w}} = 0$$
$$A_z = \mathrm{i} \begin{pmatrix} |\phi|^2 & \pm\phi_x \\ \bar{\phi}_x & -|\phi|^2 \end{pmatrix} \qquad 2A_{\tilde{z}} = \pm\mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.11}$$

Here the upper (lower) sign corresponds to $G = SU(2)$ (or $SU(1, 1)$). The SDYM equations become $\mathrm{i}\phi_z = -\phi_{xx} \mp 2|\phi|^2\phi$ which is the nonlinear Schrödinger equation with an attractive (respectively repulsive) self-interaction [6].

3. SDYM and self-dual gravity

Let $\mathcal{M}$ be a four-dimensional (4D) complex manifold (for example the complexification of some real slice $\mathcal{M}_{\mathbb{R}}$) and let g be a holomorphic metric on $\mathcal{M}$ (for example the complexification of a real metric on $\mathcal{M}_{\mathbb{R}}$). The following theorem states that the self-duality equations on the curvature can be expressed in terms of the consistency condition for a Lax pair of vector fields.

Theorem 3.1. (Mason and Newman [5]). Let $V_a = (W, \tilde{W}, Z, \tilde{Z})$ be four independent holomorphic vector fields on a four-dimensional complex manifold $\mathcal{M}$ and let ν be a non-zero holomorphic 4-form. Put

$$L = W - \lambda\tilde{Z} \qquad M = Z - \lambda\tilde{W}. \tag{3.12}$$

Suppose that for every $\lambda \in \mathbb{C}P^1$

$$[L, M] = 0 \tag{3.13}$$

$$\mathcal{L}_L \nu = -\mathcal{L}_M \nu = 0. \tag{3.14}$$

Here $\mathcal{L}_V$ denotes a Lie derivative. Then $\sigma_a = f^{-1} V_a$, where $f^2 = \nu(W, \tilde{W}, Z, \tilde{Z})$, is a normalized null-tetrad for a half-flat metric (i.e. with vanishing Ricci tensor and self-dual Weyl tensor). Every half-flat metric arises in this way.

The covariant metric is conveniently expressed in terms of the dual frame e_{V_a}

$$g = f^2(e_W \odot e_{\tilde{W}} - e_Z \odot e_{\tilde{Z}}) \tag{3.15}$$

where

$$e_W = f^{-2}\nu(\ldots, \tilde{W}, Z, \tilde{Z}) \qquad e_{\tilde{W}} = f^{-2}\nu(W, \ldots, Z, \tilde{Z})$$
$$e_Z = f^{-2}\nu(W, \tilde{W}, \ldots, \tilde{Z}) \qquad e_{\tilde{Z}} = f^{-2}\nu(W, \tilde{W}, Z, \ldots). \tag{3.16}$$

The operators L and M determine a basis of anti-self-dual (ASD) two-forms on $\mathcal{M}$

$$\alpha = f^2 e_W \wedge e_Z \qquad \omega = f^2(e_W \wedge e_{\tilde{W}} - e_Z \wedge e_{\tilde{Z}}) \qquad \tilde{\alpha} = f^2 e_{\tilde{W}} \wedge e_{\tilde{Z}}. \tag{3.17}$$

We note that $-\mathrm{i}(\alpha - \tilde{\alpha})$, $\mathrm{i}\omega$ and $\alpha + \tilde{\alpha}$ are non-degenerate symplectic forms, which (together with three compatible complex structures) endow $\mathcal{M}$ with a complexified hyper-Kähler structure.

4. Self-dual metrics on $\mathbb{R}^2 \times \Sigma$

We connect the self-duality equations on a Yang–Mills field and those on a 4D metric by considering gauge potentials that take values in a Lie algebra of vector fields on some manifold. Theorem (3.1) reveals one such connection: W, $\tilde{W}$, Z and $\tilde{Z}$ are generators of the group of volume-preserving (holomorphic) diffeomorphisms of $(\mathcal{M}, \nu)$. We make the identification: $W = D_w$, $\tilde{W} = D_{\tilde{w}}$, $Z = D_z$ and $\tilde{Z} = D_{\tilde{z}}$. By comparing (3.13) with (2.7), we see that the half-flat equation is a reduction of the SDYM with this gauge group by translations along the four coordinate vectors ∂_w, $\partial_{\tilde{w}}$, ∂_z and $\partial_{\tilde{z}}$.

In order to understand the relationship with 2D integrable systems, we look at this in a slightly different way. Let (Σ, Ω_Σ) be a 2D symplectic manifold and let $\mathrm{SDiff}(\Sigma)$ be the group of canonical transformations of Σ. Consider the SDYM equations with the gauge group G, where G is the subgroup of $\mathrm{SDiff}(\Sigma)$. We can represent the components of the connection form of D by Hamiltonian vector fields and hence by Hamiltonians on Σ depending also on the coordinates on $\mathbb{M}$

$$W = \partial_w - X_{H_w} \qquad \tilde{W} = \partial_{\tilde{w}} - X_{H_{\tilde{w}}} \qquad Z = \partial_z - X_{H_z} \qquad \tilde{Z} = \partial_{\tilde{z}} - X_{H_{\tilde{z}}} \tag{4.18}$$

where X_{H_μ} denotes the Hamiltonian vector field corresponding to A_μ with Hamiltonian H_μ.

Now we suppose that D is invariant under two translations. The reduced Lax pair will then descend to $\mathbb{R}^2 \times \Sigma$ and give rise to a half-flat metric. This requires that the gauge group is a subgroup of the canonical transformations of Σ. Although it has been observed that $\mathrm{SDiff}(\Sigma) \approx SL(\infty)$, it seems that $SL(n, \mathbb{C})$ is a subgroup of such defined $SL(\infty)$ only for $n = 2$ [7]. In this case we can take the linear action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$ or a Möbius action of $SU(2)$ and $SU(1, 1)$ on $\mathbb{C}P^1$ or D (the Poincaré disc), respectively. We shall restrict ourselves to real vector fields, which will imply that our SD metrics will have ultra-hyperbolic signature (Euclidean examples can be obtained in a similar way).

To be more explicit we write down the Hamiltonian† corresponding to the matrix

$$A_\mu = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in L\,SL(2,\mathbb{C}).$$

In the three cases we have

$$\Sigma = R^2 \qquad \Omega_\Sigma = \mathrm{d}m \wedge \mathrm{d}n \qquad H_\mu = \left(\frac{bn^2}{2} + amn - \frac{cm^2}{2}\right) \tag{4.19}$$

$$\Sigma = \mathbb{C}P^1 \qquad \Omega_\Sigma = \frac{\mathrm{i}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}}{(1+\xi\bar{\xi})^2} \qquad H_\mu = -\mathrm{i}\frac{\xi\bar{b} - \bar{\xi}b + 2a}{1+\xi\bar{\xi}} \tag{4.20}$$

$$\Sigma = D \qquad \Omega_\Sigma = \frac{\mathrm{i}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}}{(1-\xi\bar{\xi})^2} \qquad H_\mu = -\mathrm{i}\frac{\xi\bar{b} - \bar{\xi}b - 2a}{1-\xi\bar{\xi}}. \tag{4.21}$$

The form of the null-tetrad (3.16) and the hyper-Kähler structure (3.17) obtained after the 2D reductions of SDYM is as follows.

(i) H_1, $(X = \partial_w, Y = \partial_{\tilde{w}})$, $\nu = \mathrm{d}z \wedge \mathrm{d}\tilde{z} \wedge \Omega_\Sigma$

$$f^2 = \nu(W, \tilde{W}, Z, \tilde{Z}) = \Omega_\Sigma(W, \tilde{W}) = \{H_w, H_{\tilde{w}}\} = F_{w\tilde{w}}. \tag{4.22}$$

In the last formula $F_{w\tilde{w}}$ is a function rather than a matrix. This follows from the identification (via (4.19)–(4.21)) of (2×2) matrices in the Lie algebra of $SL(2,\mathbb{C})$ as Hamiltonians. Let d_Σ stand for the exterior derivative on Σ.

$$\begin{aligned}
e_W &= f^{-2}(\Omega_\Sigma(\tilde{W}, Z)\,\mathrm{d}z + \Omega_\Sigma(\tilde{W}, \tilde{Z})\,\mathrm{d}\tilde{z} + \Omega_\Sigma(\ldots, \tilde{W})) \\
&= f^{-2}(\{H_{\tilde{w}}, H_z\}\,\mathrm{d}z + \{H_{\tilde{w}}, H_{\tilde{z}}\}\,\mathrm{d}\tilde{z} - \mathrm{d}_\Sigma H_{\tilde{w}}) \\
e_{\tilde{W}} &= f^{-2}(\{H_w, H_z\}\,\mathrm{d}z - \{H_w, H_{\tilde{z}}\}\,\mathrm{d}\tilde{z} + \mathrm{d}_\Sigma H_w) \\
e_Z &= \mathrm{d}z \\
e_{\tilde{Z}} &= \mathrm{d}\tilde{z} \\
\alpha &= -\{H_{\tilde{w}}, H_{\tilde{z}}\}\,\mathrm{d}z \wedge \mathrm{d}\tilde{z} - \mathrm{d}_\Sigma H_{\tilde{w}} \wedge \mathrm{d}z \\
\omega &= (\{H_z, H_{\tilde{z}}\} - \{H_w, H_{\tilde{w}}\})\,\mathrm{d}z \wedge \mathrm{d}\tilde{z} + \Omega_\Sigma + \mathrm{d}_\Sigma H_z \wedge \mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde{z}} \wedge \mathrm{d}\tilde{z} \\
\tilde{\alpha} &= \{H_w, H_z\}\,\mathrm{d}z \wedge \mathrm{d}\tilde{z} + \mathrm{d}_\Sigma H_w \wedge \mathrm{d}\tilde{z}.
\end{aligned} \tag{4.23}$$

The gauge freedom is used to set $A_{\tilde{z}}$ (and hence $H_{\tilde{z}}$) to zero.

$$\begin{aligned}
\mathrm{d}s^2 = \frac{1}{\{H_w, H_{\tilde{w}}\}}&(-(\{H_{\tilde{w}}, H_z\}\{H_w, H_z\})\,\mathrm{d}z^2 - \{H_w, H_{\tilde{w}}\}^2\,\mathrm{d}z\,\mathrm{d}\tilde{z} - (\partial_\xi H_w \partial_\xi H_{\tilde{w}})\,\mathrm{d}\xi^2 \\
&-(\partial_{\bar{\xi}} H_w \partial_{\bar{\xi}} H_{\tilde{w}})\,\mathrm{d}\bar{\xi}^2 - ((\partial_\xi H_{\tilde{w}} \partial_{\bar{\xi}} H_w) + (\partial_{\bar{\xi}} H_{\tilde{w}} \partial_\xi H_w))\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi} + (\partial_\xi H_{\tilde{w}}\{H_w, H_z\} \\
&+\partial_\xi H_w\{H_{\tilde{w}}, H_z\})\,\mathrm{d}z\,\mathrm{d}\xi + (\partial_{\bar{\xi}} H_{\tilde{w}}\{H_w, H_z\} + \partial_{\bar{\xi}} H_w\{H_{\tilde{w}}, H_z\})\,\mathrm{d}z\,\mathrm{d}\bar{\xi}).
\end{aligned}$$

(ii) H_2 $(X = \partial_w - \partial_{\tilde{w}}, Y = \partial_{\tilde{z}})$, $\nu = \mathrm{d}x \wedge \mathrm{d}z \wedge \Omega_\Sigma$

$$f^2 = \{H_w - H_{\tilde{w}}, H_{\tilde{z}}\} = F_{w\tilde{z}} \tag{4.24}$$

$$\begin{aligned}
e_W &= f^{-2}(\{H_{\tilde{z}}, H_{\tilde{w}}\}\,\mathrm{d}x + \{H_{\tilde{z}}, H_z\}\,\mathrm{d}z - \mathrm{d}_\Sigma H_{\tilde{z}}) \\
e_{\tilde{W}} &= f^{-2}(-\{H_{\tilde{z}}, H_w\}\,\mathrm{d}x - \{H_{\tilde{z}}, H_z\}\,\mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde{z}}) \\
e_Z &= \mathrm{d}z \\
e_{\tilde{Z}} &= f^{-2}(\{H_w, H_{\tilde{w}}\}\,\mathrm{d}x + \{H_w - H_{\tilde{w}}, H_z\}\,\mathrm{d}z - \mathrm{d}_\Sigma(H_w - H_{\tilde{w}})) \\
\alpha &= \{H_{\tilde{z}}, H_{\tilde{w}}\}\,\mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde{z}} \wedge \mathrm{d}z
\end{aligned} \tag{4.25}$$

† We only require the representation of A_μ by volume-preserving vector fields on Σ; Hamiltonians are defined up to the addition of a function of the (residual) space variables, but different choices of such functions do not change the metric.

$$\omega = (\{H_w, H_z\} - \{H_{\tilde{w}}, H_{\tilde{z}}\})\, \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde{z}} \wedge \mathrm{d}x - \mathrm{d}_\Sigma (H_w - H_{\tilde{w}}) \wedge \mathrm{d}z$$
$$\tilde{\alpha} = \{H_w, H_z\}\, \mathrm{d}x \wedge \mathrm{d}z + \Omega_\Sigma + \mathrm{d}_\Sigma H_w \wedge \mathrm{d}x - \mathrm{d}_\Sigma H_z \wedge \mathrm{d}z.$$

We can perform a further gauge transformation to set $H_{\tilde{w}} = 0$ in which case

$$\begin{aligned}
\mathrm{d}s^2 = &-\left(\frac{\{H_{\tilde{z}}, H_z\}^2}{\{H_w, H_{\tilde{z}}\}} + \{H_w, H_z\}\right) \mathrm{d}z^2 - \frac{(\partial_\xi H_{\tilde{z}})^2}{\{H_w, H_{\tilde{z}}\}}\, \mathrm{d}\xi^2 - \frac{(\partial_{\bar{\xi}} H_{\tilde{z}})^2}{\{H_w, H_{\tilde{z}}\}}\, \mathrm{d}\bar{\xi}^2 \\
&+ \left(\partial_\xi H_w + 2\frac{\{H_{\tilde{z}}, H_z\}}{\{H_w, H_{\tilde{z}}\}} \partial_\xi H_{\tilde{z}}\right) \mathrm{d}z\, \mathrm{d}\xi + \left(\partial_{\bar{\xi}} H_w + 2\frac{\{H_{\tilde{z}}, H_z\}}{\{H_w, H_{\tilde{z}}\}} \partial_{\bar{\xi}} H_{\tilde{z}}\right) \mathrm{d}z\, \mathrm{d}\bar{\xi} \\
&- \frac{2\partial_\xi H_{\tilde{z}} \partial_{\bar{\xi}} H_{\tilde{z}}}{\{H_w, H_{\tilde{z}}\}}\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi} + \{H_{\tilde{z}}, H_z\}\, \mathrm{d}x\, \mathrm{d}z - \partial_\xi H_{\tilde{z}}\, \mathrm{d}x\, \mathrm{d}\xi - \partial_{\bar{\xi}} H_{\tilde{z}}\, \mathrm{d}x\, \mathrm{d}\bar{\xi}.
\end{aligned}$$

Reductions by $X = \partial_w, Y = \partial_z$ are not considered because the resulting metric turns out to be degenerate everywhere as a direct consequence of the SDYM equations. Equation (2.4) now becomes $[X_{H_w}, X_{H_z}] = 0$ which, in the case of finite-dimensional sub-algebras of LSDiff(Σ), implies linear dependence of X_{H_w} and X_{H_z}.

The construction naturally applies to the complex four-manifolds. We start from the SDYM equations on $\mathbb{C}^4$ with gauge group $SL(2, \mathbb{C})$. Then we perform one of the possible reductions to $\mathbb{C}^2$. Let $\Sigma^2_{\mathbb{C}}$ be a two-dimensional complex manifold, for example $\mathbb{C}P^1 \times \mathbb{C}P^{1*}$. $SL(2, \mathbb{C})$ acts on one Riemann sphere by a Möbius transformation, and on the other by the inverse

$$(\xi, \tilde{\xi}) \longrightarrow \left(\frac{A\xi + B}{C\xi + D}, \frac{D\tilde{\xi} - C}{-B\tilde{\xi} + A}\right).$$

Here ξ and $\tilde{\xi}$ are independent complex coordinates on $\mathbb{C}P^1$ and $\mathbb{C}P^{1*}$. The action preserves the symplectic form $\Omega_{\Sigma_{\mathbb{C}}} = (1 + \xi\tilde{\xi})^{-2}(\mathrm{d}\xi \wedge \mathrm{d}\tilde{\xi})$ defined on the complement of $1 + \xi\tilde{\xi} = 0$. All results of this section may be extended to the complex case by replacing $\bar{\xi}$ by the independent coordinate $\tilde{\xi}$.

4.1. Solitonic metrics

We can now establish the connection between the integrable systems reviewed in section 2 and self-dual vacuum metrics. We do so by expressing the Hamiltonians above in terms of solutions to various soliton equations. From a given solution of a 2D nonlinear equation we can generate a null-tetrad (3.16).

(1) Nonlinear Schrödinger equation.

$$\begin{aligned}
W &= \partial_x + (\bar{\phi}\xi^2 + \phi)\partial_\xi + (\phi\bar{\xi}^2 + \bar{\phi})\partial_{\bar{\xi}} \\
\tilde{W} &= \partial_x \\
\tilde{Z} &= -\mathrm{i}\xi\partial_\xi + \mathrm{i}\bar{\xi}\partial_{\bar{\xi}} \\
Z &= \partial_z + \mathrm{i}(-\bar{\phi}_x\xi^2 + 2|\phi|^2\xi + \phi_x)\partial_\xi - \mathrm{i}(-\phi_x\bar{\xi}^2 + 2|\phi|^2\bar{\xi} + \bar{\phi}_x)\partial_{\bar{\xi}} \\
f^2 &= \frac{2\,\mathrm{Re}(\bar{\xi}\phi)}{1 + |\xi|^2}.
\end{aligned}$$

(2) Korteweg de Vries equation.

$$\begin{aligned}
W &= \partial_x + (qm + n)\partial_m + (bm - qn)\partial_n \\
\tilde{W} &= \partial_x \\
\tilde{Z} &= m\partial_n
\end{aligned}$$

$$
\begin{aligned}
Z &= \partial_z + \left(\frac{b_x}{2}m - q_x n\right)\partial_m + \left(wm - \frac{b_x}{2}n\right)\partial_n \\
f^2 &= -m(q+mn)
\end{aligned}
$$

where $b = q_x - q^2$ and $4w = q_{xxx} - 4qq_{xx} - 2{q_x}^2 + 4q^2q_x$.

(3) SG; elliptic case.

$$
\begin{aligned}
W &= \partial_x + \tfrac{1}{4}(\phi_t m - 2\cos(\phi/2)n)\partial_m + \tfrac{1}{4}(-\phi_t n - 2\cos(\phi/2)m)\partial_n \\
\tilde{W} &= \partial_x + \tfrac{1}{4}(\phi_t m + 2\cos(\phi/2)n)\partial_m + \tfrac{1}{4}(-\phi_t n + 2\cos(\phi/2)m)\partial_n \\
\tilde{Z} &= \partial_t + \tfrac{1}{4}(-\phi_x m - 2\sin(\phi/2)n)\partial_m + \tfrac{1}{4}(\phi_x n - 2\sin(\phi/2)m)\partial_n \\
Z &= \partial_t + \tfrac{1}{4}(-\phi_x m + 2\sin(\phi/2)n)\partial_m + \tfrac{1}{4}(\phi_x n + 2\sin(\phi/2)m)\partial_n \\
f^2 &= (\sin\phi)mn.
\end{aligned}
$$

(4) SG; hyperbolic case.

$$
\begin{aligned}
W &= (-\mathrm{i}\xi^2\,\mathrm{e}^{-\mathrm{i}\phi} + \mathrm{i}\mathrm{e}^{\mathrm{i}\phi})\partial_\xi + (\mathrm{i}\bar{\xi}^2\,\mathrm{e}^{\mathrm{i}\phi} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi})\partial_{\bar{\xi}} \\
\tilde{W} &= (-\mathrm{i}\xi^2 + \mathrm{i})\partial_\xi + (\mathrm{i}\bar{\xi}^2 - \mathrm{i})\partial_{\bar{\xi}} \\
\tilde{Z} &= \partial_{\tilde{z}} \\
Z &= \partial_z - \mathrm{i}(\partial_{\tilde{z}}\phi)\xi\partial_\xi + \mathrm{i}(\partial_{\tilde{z}}\phi)\bar{\xi}\partial_{\bar{\xi}} \\
f^2 &= \frac{4\sin\phi(|\xi|^2 - 1)}{|\xi|^2 + 1}.
\end{aligned}
$$

Put $\mathrm{d}_A\xi = \mathrm{d}\xi + \mathrm{i}\xi\partial_{\tilde{z}}\phi\,\mathrm{d}z$. Then we have

$$
\begin{aligned}
\mathrm{d}s^2 = \frac{1}{1+\xi\bar{\xi}}&([(1-\bar{\xi}^2)^2\cot\phi + \mathrm{i}(1-\bar{\xi}^4)]\,\mathrm{d}_A\xi\otimes\mathrm{d}_A\xi + 2\sin\phi\,\mathrm{d}z\otimes\mathrm{d}\tilde{z} \\
&+(\cot\phi(1-\bar{\xi}^2)(1-\xi^2) + \mathrm{i}[(1+\bar{\xi}^2)(1-\xi^2) \\
&-(1-\bar{\xi}^2)(1+\xi^2)])\,\mathrm{d}_A\xi\otimes\overline{\mathrm{d}_A\xi} + [(1-\xi^2)^2\cot\phi - \mathrm{i}(1-\xi^4)]\,\overline{\mathrm{d}_A\xi}\otimes\overline{\mathrm{d}_A\xi}).
\end{aligned}
\tag{4.26}
$$

If one takes a solution describing the interaction of a half kink and a half anti-kink (two topological solitons travelling in $z-\tilde{z}$ direction and increasing from 0 to π as $z+\tilde{z}$ goes from $-\infty$ to ∞) then the singularity in $\sin\phi = 0$ may be absorbed by a conformal transformation of $z+\tilde{z}$ [3].

From the Yang–Mills point of view, the solutions that we have obtained are metrics on the total space of $\mathcal{E}$, the Σ-bundle associated to the Yang–Mills bundle. Therefore, it is of interest to consider the effect of gauge transformations. First, notice that diffeomorphisms of $\mathbb{R}^2\times\Sigma$ given by

$$
x^a \longrightarrow x^a + \epsilon X_F(x^a) \tag{4.27}
$$

yield $H_\mu \longrightarrow H_\mu + \epsilon(\{H_\mu, F\} + \partial_\mu F)$ which is an infinitesimal form of the full gauge transformation (2.3). Here μ is an index on $\mathbb{M}$, whereas a is an index on $\mathcal{M} = \mathbb{R}^2\times\Sigma$. The vector field X_F is Hamiltonian with respect to Ω_Σ, with Hamiltonian $F = F(x^a)$.

If (4.27) preserves the Kähler structure of Σ then H_μ transforms under (a real form of) $SL(2,\mathbb{C})$ and therefore our construction remains 'invariant'.

5. Final remarks

5.1. The relationship between the twistor correspondences

To finish, we explain how our construction ties in with the twistor correspondences for the self-duality equations. We consider only the complex case of the SDYM equations with two commuting symmetries X, Y. The $SL(2,\mathbb{C})$ SDYM connection defines, by the Ward construction [11], a holomorphic vector bundle over the (non-deformed) twistor space, $E_W \to \mathcal{P}$. It is convenient† to use the bundle $\mathcal{E}_W^5$—associated to E_W by the representation of $SL(2,\mathbb{C})$ as holomorphic canonical transformations of the complex symplectic manifold $\Sigma^2_{\mathbb{C}}$.

On the other hand, the SD vacuum metric corresponds to a deformed twistor space $\mathcal{P}_{\mathcal{M}}$, [10]. In this paper we have explained how the quotient of $\mathcal{E}$ by lifts of X, Y is, by theorem (3.1), equipped with a half-flat metric . To give a more complete picture we can obtain the deformed twistor space directly from $\mathcal{E}_W^5$ and show that this is the twistor space of $\mathcal{M}$. Consider the following chain of correspondences:

$$\begin{array}{ccccccccccc} & \mathcal{E}_W^5 & & & \mathcal{F} & & \mathcal{E} & & \mathcal{F}_{\mathcal{M}} & & \\ \swarrow & & \searrow & \swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow & \\ \mathcal{P}_{\mathcal{M}} & & & \mathcal{P} & & \mathbb{C}^4 & & \mathcal{M} & & & \mathcal{P}_{\mathcal{M}}. \end{array}$$

Here $\mathcal{F}$ and $\mathcal{F}_{\mathcal{M}}$ are the standard projective spin bundles fibred over $\mathbb{C}^4$ and $\mathcal{M}$, respectively. The space $\mathcal{F}^7_{\mathcal{E}}$, the pullback of the spin bundle $\mathcal{F}$ to the total space of the bundle $\mathcal{E}$, fibres over all the spaces in the above diagram. Taking the quotient by lifts of X, Y we project $\mathcal{F}^7_{\mathcal{E}}$ to $\mathcal{F}_{\mathcal{M}}$. Taking the quotient by the twistor distribution, $\mathcal{F}^7_{\mathcal{E}}$ also projects to the Ward bundle $\mathcal{E}_W^5$. By definition it projects to $\mathcal{E}$ and it could equivalently have been defined as the pullback of $\mathcal{E}$ to $\mathcal{F}$. The compatibility of these projections is a consequence of the commutativity of the diagram

$$\begin{array}{rcl} \mathbb{C}P^1 \times \mathbb{C}^4 \times \Sigma^2_{\mathbb{C}} = \mathcal{F}^7_{\mathcal{E}} & \xrightarrow{(X,Y)} & \mathcal{F}^5_{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathcal{E}^5_W & \xrightarrow{(X,Y)} & \mathcal{P}_{\mathcal{M}} \end{array} \tag{5.28}$$

which follows from the integrability the distribution spanned by (lifts of) X, Y, L, M, and from the fact that (X, Y) commute with (L, M).

5.2. Global issues

In order to obtain a compact space one might attempt the following:

- choose the gauge group to be $SU(2)$ so that the fibre space is compact, and
- compactify $\mathbb{R}^2$ after the reduction.

We restrict the rate of decay of A_μ by the requirement that A_μ should be smoothly extendible to S^2 in the split signature case. Other possibilities are to restrict to the class of rapidly decreasing soliton solutions of the corresponding integrable equation. If we have reduced from a Euclidean signature solution to the SDYM equations, then it is more natural to compactify $\mathbb{R}^2$ in such a way as to obtain a Riemann surface of genus greater than one as it is only for such compactifications that one can find the existence of non-trivial solutions, [4].

† The diagram (5.28) describes also the general case of $G = \mathrm{SDiff}(\Sigma^2_{\mathbb{C}})$. For this we work with $\mathcal{E}^5_W$ rather than the principal Ward bundle, since the latter has infinite-dimensional fibres. The notation is such that the upper index of a space stands for the complex dimension of that space.

However, we still have singularities in the metrics corresponding to (4.23) and (4.25), even if we can eliminate those from the Yang–Mills connection. We are left with singularities associated with sets on which the tetrad becomes linearly dependent. This reduces to the proportionality (or vanishing) of the Higgs fields on Σ, which generically occurs on a real co-dimension one subset of each fibre (and hence co-dimension one in the total space). In the above formulae this set is given by the vanishing of f. The Weyl curvature C_{abcd} blows up as f goes to zero. Calculation of curvature invariants shows that these lead to genuine singularities that cannot be eliminated by a change of frame or coordinates. For example

$$C_{abcd}C^{abcd} = \sum_{i=-3}^{3} C_i f^{2i}$$

where $C_i = C_i(x^a)$ are generally non-vanishing regular functions on $\mathcal{M}$, which explicitly depend on the Yang–Mills curvature $F_{\mu\nu}$ and (derivatives of) Hamiltonians (4.19)–(4.21). Those singularities appear (for purely topological reasons) because each vector in the tetrad $(W, \tilde{W}, Z, \tilde{Z})$ has at least one zero, when restricted to $\Sigma = S^2$.

One can also obtain Euclidean metrics as above by using reductions of the SDYM equations from Euclidean space, but we will still be unable to avoid these same co-dimension one singularities.

5.3. Other reductions

We have focused in this article on the familiar $(1+1)$ soliton equations. However, it is clear from the discussions of sections 3 and 4 that the construction will extend to any symmetry reduction of the SDYM equations to systems in two dimensions with gauge group contained in $SL(2, \mathbb{C})$, in particular when the symmetry imposed consists of two translations as for the Euclidean signature examples mentioned previously. However, one can also use the same device to embed examples using any other 2D symmetry subgroup of the conformal group. In particular, with cylindrical symmetry, one obtains the Ernst equations (the two symmetry reduction of the full, non-self-dual 4D Einstein vacuum equations) and this can similarly be embedded into the self-dual (but not vacuum) equations.

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